Akhil Mathew.
THH and crystalline cohomology II.
Drinfeld seminar.
2 November 2017.

A a commutative ring.
B smooth A-algebra.
$(\Omega^*_{B/A}, d)$.

Consider $A = k$, a perfect field of char. $p > 0$.

Consider $B \longmapsto (\Omega^\bullet_{B/k}, d)$ as a cohomology theory for smooth schemes over k.

Problem: it takes values in k-vector spaces.
Can we lift it to char. 0?

Crystalline cohomology

$$\begin{tikzcd} & & & p\text{-complete } \mathbb{E}_\infty\text{-alg}/W(k) \arrow[d] \\ \text{smooth } k\text{-algebras} \arrow[r, "\Omega^\bullet_{-/k}"] \arrow[urrr, bend left, "\mathrm{crys}"] & \mathrm{cdgas}/k \arrow[rr] & & \mathbb{E}_\infty\text{-alg}/k \end{tikzcd}$$

$$\mathrm{crys}(B)/p \simeq \Omega^\bullet_{B/k}.$$

If B lifts to $\tilde{B}/W$ which is p-adically complete, smooth, then $\mathrm{crys}(B) \simeq \Omega^\bullet_{\tilde{B}/W}$ (p-completely).

If B is a smooth k-algebra, the construction $W\Omega^\bullet_B$, the de Rham Witt complex, models $\mathrm{crys}(B)$.
But, it is much bigger than $\Omega^\bullet_{\tilde{B}/W}$.
If $\tilde{B}$ also has a lift φ of Frobenius, then there is a well-defined q-iso $(\Omega^\bullet_{\tilde{B}/W(k)}, d) \simeq W\Omega^\bullet_B$.
p-completely again.

We will not get a theory in dgas when working in mixed characteristic.

In any case, going to leave the smooth case.

Derived de Rham and crystalline cohomologies

Dold-Puppe. Non-additive derived functors.

Ex. R a commutative ring.

$$F: \mathrm{Proj}(R)^{\omega} \longrightarrow D(R)_{\geq 0}.$$

Get $F: D(R)_{\geq 0} \longrightarrow D(R)_{\geq 0}$. Assume F lands in $D(R)^{\omega}$. Take $X \in D(R)_{\geq 0}$. Represent X by a simplicial complex $X_\bullet$ by Dold-Kan. We can assume that $X_\bullet \in s\mathrm{Proj}(R)$. Just extend along $\mathrm{Proj}(R)^{\omega} \longrightarrow \mathrm{Proj}(R)$ and then take simplicial geo. realizations. Etc.

Let $\mathcal{C}$ be an ∞-cat with sifted colimits. Then,

$$\mathrm{Fun}(\mathrm{Proj}_R^{\omega}, \mathcal{C}) \simeq \mathrm{Fun}^{\text{sifted}}(D(R)_{\geq 0}, \mathcal{C}).$$

Exs. a) $\mathrm{Sym}^r(M[1]) \simeq (\Lambda^r M)[r]$, $M \in D(R)_{\geq 0}$.

$\left((M^{\otimes r})_{h\Sigma_r} \text{ is typically too big. in char. } p \right)$

b) $\mathrm{Sym}^r(M[2]) \simeq \Gamma^r(M)[2r]$.

Γ^r: Divided powers.

Cotangent complex.

Derived functors of $\Omega^1_{-/k}$. $(L_{-/k})$

$SCR_k \leftarrow$ built out of f.g. polynomial k-algebras by freely adding sifted colims.

$$\mathrm{Fun}\left(\mathrm{CAlg}_k^{\heartsuit,\,\omega\text{-poly}}, \mathcal{C}\right) \simeq \mathrm{Fun}^{\text{sifted}}(SCR_k, \mathcal{C}).$$

Key facts.

1) $L_{B/k} \simeq \Omega^1_{B/k}$ for B/k smooth.

2) $A \to B \to C$

$$L_{B/A} \otimes_B C \to L_{C/A} \to L_{C/B},$$

cofiber sequence.

3) $L_{\mathbb{Z}[x] \to \mathbb{Z}} \simeq \mathbb{Z}[1]$.

4) R = any, $r \in R$ a non-zero divisor,

$$L_{R \to R/(r)} \simeq R/(r)[1].$$

Base change from (3).

So, controllable for complete intersections.

5) If R is a perfect $\mathbb{F}_p$-algebra,

$$L_{R/\mathbb{F}_p} \simeq 0.$$

Derived de Rham

$$\mathrm{CAlg}_A^{\infty,\omega\text{-poly}} \xrightarrow{\Omega^\bullet_{A/k}} \mathrm{CAlg}_A^{E_\infty}$$

Extend to $\quad dR_{-/k} : \mathrm{SCR}_A \xrightarrow{dR} \mathrm{CAlg}_A^{E_\infty}.$

Ex. (1) In char. 0, $dR_{\mathbb{R}/\mathbb{Q}} \simeq \mathbb{Q}$ by calculating $dR_{\mathbb{Q}[x]/\mathbb{Q}}$.

(2) In char. p, behaves better thanks to Cartier and the conjugate filtration. Recall that if B/A is a smooth algebra, A char $p>0$,

$$H^i(\Omega^\bullet_{B/A}, d) \cong \Omega^i_{B^{(1)}/A} \qquad (\text{Cartier}).$$

So, if P/A is a polynomial algebra, $(\Omega^\bullet_{P/A}, d)$ has a Postnikov filtration with successive quotients

$$\Lambda^i \Omega_{P/A}[-i].$$

CONJUGATE FILTRATION.

So, there is an increasing exhaustive filtration on $dR_{B/A}$ with successive quotients

$$B^{(1)}, \quad L_{B^{(1)}/A}[-1] \quad \Lambda^2 L_{B^{(1)}/A}[-2], \ldots$$

Exhaustive means the colimit recovers $dR_{B/A}$.

So, if B/A is smooth,

$$dR_{B/A} \simeq \Omega^\bullet_{B/A}$$

(in char p).

Ex. $\mathbb{Z}[x] \longrightarrow \mathbb{Z}$

$x \longmapsto 0.$

Let's complete $\left(dR_{\mathbb{Z}/\mathbb{Z}[x]}\right)^\wedge_p$. If we rationalize before completion, we get zero.

$$\left(dR_{\mathbb{Z}/\mathbb{Z}[x]}\right)^\wedge_p \simeq \Gamma_{\mathbb{Z}_p}(x), \quad |x|=0.$$

Divided power algebra.

So, surprisingly, this is discrete. (Compare $L_{\mathbb{Z}[x]\to\mathbb{Z}} \simeq \mathbb{Z}[1]$.)

Ex. Reduce the previous mod p getting

$$dR_{\mathbb{F}_p/\mathbb{F}_p[x]} \simeq \Gamma_{\mathbb{F}_p}(x).$$

Note Frobenius.

$A = \mathbb{F}_p[x]$

$B = \mathbb{F}_p$

$$B^{(1)} \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p[x]} \mathbb{F}_p[x] \simeq \frac{\mathbb{F}_p[x]}{(x^p)}$$

$L_{B^{(1)}/A} \simeq B^{(1)}[1]$ using ci formula.

$$\Lambda^i L_{B^{(1)}/A} \simeq \Lambda^i\left(B^{(1)}[1]\right)$$
$$\simeq \left(\Gamma^i B^{(1)}\right)[i].$$

Rem. Can also do a decreasing filtration and Hodge complete.

Derived crystalline cohomology

k perfect field of char. p.

$$\begin{tikzcd}
\mathrm{Poly}_k^{\omega} \arrow[r, "\mathrm{crys}"] \arrow[d] & \mathrm{CAlg}_{W(k)}^{\mathrm{Fil}} \\
\mathrm{SCR}_k \arrow[ur, "\text{crys, or dcrys.}"']
\end{tikzcd}$$

Derived crystalline cohomology.

<u>Rem</u> (From Rodin).

<u>Rem</u>. $\mathrm{dcrys}(B) \simeq dR_{B/W(k)} \otimes_{dR_{k/W(k)}} W(k)$. [All p-complete.]

We have

$$dR_{k/W(k)} \simeq W(k) \otimes_{\mathbb{Z}[x]} \Gamma_{\mathbb{Z}_p}(x)$$

Ring that classifies divided power structures on (p).

$$x \mapsto p.$$

Lots of torsion. But, unique map

$$dR_{k/W(k)} \longrightarrow W(k)$$

sending the divided powers to divided powers.

BMS filtration.

R smooth$/k$, perfect field of char $p>0$.

R_{perf} = colim along Frobenius.

$$R \longrightarrow R_{perf} \rightrightarrows R_{perf} \otimes R_{perf} \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots$$

R_{perf}: Perfect

$R_{perf} \otimes R_{perf}$: Not perfect, but regular semiperfect.

= Perfect modulo a regular sequence.

(Zariski locally.)

$R = \mathbb{F}_p[x]$

$R_{perf} \cong \mathbb{F}_p[x^{1/p^\infty}]$.

$$R_{perf} \otimes_R R_{perf} \cong \frac{\mathbb{F}_p[x^{1/p^\infty}] \otimes_{\mathbb{F}_p} [y^{1/p^\infty}]}{(x-y)}$$

$$\cong \mathbb{F}_p[u^{1/p^\infty}] \otimes \mathbb{F}_p[t^{1/p^\infty}]/(t).$$

<u>Reference.</u> Bhatt. p-adic derived de Rham cohomology.

Berthelot.

Observation. dR, dcrys, L behave well for reg. semi-perfect.

<u>Computations</u>. (1) $R = A/(x_1,\dots,x_n)$, reg. sequence. A perfect.

Then $n=1$,

$$L_{R/\mathbb{F}_p} \simeq L_{R/A} \simeq R[1]$$

since $\quad L_{A/\mathbb{F}_p} \simeq 0$.

(2) <u>Thm</u>. R reg. semiperfect.

$$\mathrm{crys}(R) \simeq A_{crys}(R).$$

<u>Def</u>. $A_{crys}(R)$ is as follows. Write

$$R = A/I, \quad A \text{ perfect}, \quad I = (x_1,\dots,x_n).$$

$$W(A) \underbrace{\longrightarrow A \longrightarrow}_{\theta} R$$

Universal p-complete pd-thickening of R comp. with divided powers on (p).

$$\ker(\theta) = (p, [x_1], [x_2], \dots, [x_n])$$

Add divided powers to $\ker(\theta)$ compatible with those on (p). Equivalently, add divided powers to the $[x_i]$.

Ex. $R = \mathbb{F}_p[t^{1/p^\infty}]/(t)$.

$$A_{crys}(R) \cong \mathbb{Z}_p[t^{1/p^\infty}]\Big[\underbrace{\tfrac{t^p}{p}, \tfrac{t^{p^2}}{p^{p+1}}, \dots}_{\text{Or, add all terms } \frac{t^k}{k!}}\Big]^\wedge_p$$

Note $\sum_{i\geq 0} t^i$ is OK because these get more and more p-divisible.

$\sum_{i \geq 0} \frac{t^i}{i!}$ not OK.

<u>proof of theorem in this example</u>.

$$\mathrm{crys}(\mathbb{F}_p[t^{1/p^\infty}]/(t)) \simeq \widehat{dR}_{\mathbb{Z}_p[t^{1/p^\infty}]/(t)/\mathbb{Z}_p}$$

cofiber, not relative.

$$\wr\!\!\mid$$

$$\widehat{dR}_{\mathbb{Z}_p[t^{1/p^\infty}]/(t)\,/\,\mathbb{Z}_p[t^{1/p^\infty}]}$$

$$\simeq \mathbb{Z}_p[t^{1/p^\infty}] \hat{\otimes}_{\mathbb{Z}_p[x]} \Gamma_{\mathbb{Z}_p}(x)$$

$$t \longmapsto x.$$

(Bhatt -)

<u>Thm</u>: 1) If R/k is smooth, $d\mathrm{crys}$ satisfies descent along $R \longrightarrow R_{perf}$.

2) $\mathrm{crys}(R) \simeq \{ A_{crys}(R_{perf}) \rightrightarrows A_{crys}(R_{perf} \otimes R_{perf}) \substack{\to\\\to\\\to} \}$

$A_{crys}(R_{perf}) \cong W(R_{perf})$

(3) Also works for THH, TC^-, TP.

In fact, they satisfy arbitrary faithfully flat descent.

<u>Theorem</u> (for TP). If R is regular semiperfect, THH, TC^-, TP are concentrated in even degrees. Also, TP is even periodic. We also have

$$TP_0(R) \cong \hat{A}_{crys}(R),$$

completion giving top. nilpotent divided powers (or Hodge completion when there is a lift).

Take $\gamma_k(I)$, $I \subseteq A_{crys}(R)$ with $A_{crys}(R)/I \cong R$. Complete with respect to this descending filtration.

—— The BMS filtration. ——

<u>Thm</u> (BMS). R a smooth alg, $TP(R)$ has a filtration with graded pieces $gr^i\, TP(R) \simeq crys(R)[2i]$.

<u>argument</u>. $TP(R) \simeq Tot\left(TP(R_{perf}) \rightrightarrows TP(R_{perf} \otimes R_{perf}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$.

Now if R' is reg. semiperfect whose ass. graded is $\hat{A}_{crys}(R)[2i]$.

Postnikov filtration.

$$F^i TP(R) \simeq Tot\left(\tau_{\geq 2i} TP(R_{perf}) \rightrightarrows \tau_{\geq 2i} TP(R_{perf} \otimes R_{perf}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right).$$

$$F^i/F^{i+1} \simeq Tot\left(\hat{A}_{crys}(R_{perf})[2i] \rightrightarrows \hat{A}_{crys}(R_{perf} \otimes R_{perf}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

$$\simeq crys(R)[2i]$$

since $crys(R)$ is already Hodge complete, as R is smooth. (A priori limit is Hodge completed $crys(R)$.)

Hard part is calculation of THH, etc for regular semiperfect.

Drinfeld: key points are descent and deriving.

Also, regular semiperfect.